

# On the Davenport-Mahler bound

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## Abstract

We prove that the Davenport-Mahler bound holds for arbitrary graphs with vertices on the set of roots of a given univariate polynomial with complex coefficients.

## Introduction

The Davenport-Mahler bound is a lower bound for the product of the lengths of the edges on a graph whose vertices are the complex roots of a given univariate polynomial  $P \in \mathbb{C}[X]$ , under certain assumptions. Its origins are the work of Mahler ([9]), where a lower bound for the minimum separation between two roots of  $P$  in terms of the discriminant of  $P$  is given, and the work of Davenport (see [2, Proposition 8]), where for the first time a lower bound for the joint product of many different distances between roots of  $P$  (which is not simply the product of a lower bound for each distance) is obtained. Roughly speaking, this bound makes evident an interaction between the involved distances, in the sense that if some of them are very small, the rest cannot be that small.

Throughout the literature, there are different versions of this bound. We include here the one from [5, Theorem 3.1] (see also [6, 11]). We refer the reader to [10] for the definition of discriminant and Mahler measure.

**Theorem 1 (Davenport-Mahler bound)** *Let  $P \in \mathbb{C}[X]$  be a polynomial of degree  $d$ . Let  $G = (V, E)$  be a directed graph whose vertices  $\{v_1, \dots, v_k\}$  are a subset of the roots of  $P$  such that:*

1. *if  $(v_i, v_j) \in E$ , then  $|v_i| \leq |v_j|$ ,*
2.  *$G$  is acyclic,*
3. *the in-degree of any vertex is at most 1.*

Then

$$\prod_{(v_i, v_j) \in E} |v_i - v_j| \geq |\text{Disc}(P)|^{1/2} \text{M}(P)^{-(d-1)} \left( \frac{d}{\sqrt{3}} \right)^{-\#E} d^{-d/2},$$

where  $\text{Disc}(P)$  and  $\text{M}(P)$  are the discriminant and the Mahler measure of  $P$ .

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Note that when  $P$  is not a square-free polynomial, the bound becomes trivial since  $\text{Disc}(P)$  vanishes. This situation has been managed by Eigenwillig ([4, Theorem 3.9]) through the use of subdiscriminants (see [1, Section 4.2]), obtaining a generalized version of the Davenport-Mahler bound, as follows:

**Theorem 2 (Generalized Davenport-Mahler bound)** *Let  $P \in \mathbb{C}[X]$  be a polynomial of degree  $d$  with exactly  $r$  distinct complex roots. Let  $G = (V, E)$  be a directed graph whose vertices  $\{v_1, \dots, v_k\}$  are a subset of the roots of  $P$  such that:*

1. *if  $(v_i, v_j) \in E$ , then  $|v_i| \leq |v_j|$ ,*
2.  *$G$  is acyclic,*
3. *the in-degree of any vertex is at most 1.*

*Then*

$$\prod_{(v_i, v_j) \in E} |v_i - v_j| \geq |\text{sDisc}_{d-r}(P)|^{1/2} M(P)^{-(r-1)} \left(\frac{r}{\sqrt{3}}\right)^{-\#E} r^{-r/2} \left(\frac{1}{3}\right)^{\min\{d, 2d-2r\}/6}.$$

It is clear that if  $P$  is a square-free polynomial, then  $r = d$  and the bound by Eigenwillig is exactly the classical Davenport-Mahler bound.

One of the main applications of the Davenport-Mahler bound in both its classical and generalized version is its use in algorithmic complexity estimation as for instance in [3, 5, 7].

The main result in this paper is that the Generalized Davenport-Mahler bound holds for arbitrary graphs (undirected, no loops, no multiple edges) with vertices on the set of roots of  $P$ . More precisely:

**Theorem 3** *Let  $P \in \mathbb{C}[X]$  be a polynomial of degree  $d$  with exactly  $r$  distinct complex roots. Let  $G = (V, E)$  be a graph whose vertices  $\{v_1, \dots, v_k\}$  are a subset of the roots of  $P$ . Then*

$$\prod_{(v_i, v_j) \in E} |v_i - v_j| \geq |\text{sDisc}_{d-r}(P)|^{1/2} M(P)^{-(r-1)} \left(\frac{r}{\sqrt{3}}\right)^{-\#E} r^{-r/2} \left(\frac{1}{3}\right)^{\min\{d, 2d-2r\}/6}.$$

In order to prove Theorem 3, we revisit the classical proofs and the new ingredient is the use of divided differences to manage the cases where the assumptions in previous formulations do not hold.

Finally, after proving Theorem 3, we include some remarks and applications.

## 1 Proof of the results

First, we recall the definition of divided differences.

**Definition 4** *For  $f : \mathbb{C} \rightarrow \mathbb{C}$  and  $v_1, \dots, v_n \in \mathbb{C}$  with  $v_i \neq v_j$  if  $1 \leq i < j \leq n$ , the divided difference  $f[v_1, \dots, v_n] \in \mathbb{C}$  is defined inductively in  $n$  by*

$$f[v_1] = f(v_1)$$

*if  $n = 1$  and*

$$f[v_1, \dots, v_n] = \frac{f[v_1, \dots, v_{n-1}] - f[v_2, \dots, v_n]}{v_1 - v_n}$$

if  $n > 1$ .

For  $F : \mathbb{C} \rightarrow \mathbb{C}^m$  given by  $F(z) = (f_1(z), \dots, f_m(z))$  and  $v_1, \dots, v_n \in \mathbb{C}$  with  $v_i \neq v_j$  if  $1 \leq i < j \leq n$ , the divided difference  $F[v_1, \dots, v_n]$  is defined as

$$F[v_1, \dots, v_n] = (f_1[v_1, \dots, v_n], \dots, f_m[v_1, \dots, v_n]) \in \mathbb{C}^m.$$

The only properties we will use concerning divided differences are stated in the next two lemmas. We omit their proofs since they can both be easily done by induction on  $n$ . We refer the reader to [8, Chapter 6] for further properties of divided differences and their use in polynomial interpolation.

**Lemma 5** For  $F : \mathbb{C} \rightarrow \mathbb{C}^m$  and  $v_1, \dots, v_n \in \mathbb{C}$  with  $v_i \neq v_j$  if  $1 \leq i < j \leq n$ ,  $F[v_1, \dots, v_n]$  is the linear combination of  $F(v_1), \dots, F(v_n)$  given by

$$F[v_1, \dots, v_n] = \sum_{h=1}^n \left( \prod_{\substack{k=1 \\ k \neq h}}^n \frac{1}{v_h - v_k} \right) F(v_h).$$

**Lemma 6** For  $p \in \mathbb{N}_0$ ,  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by  $f(z) = z^p$ , and  $v_1, \dots, v_n \in \mathbb{C}$  with  $v_i \neq v_j$  if  $1 \leq i < j \leq n$ ,

$$f[v_1, \dots, v_n] = \begin{cases} \sum_{\substack{(t_1, \dots, t_n) \in \mathbb{N}_0^n \\ t_1 + \dots + t_n = p-n+1}} \prod_{j=1}^n v_j^{t_j} & \text{if } n \leq p+1, \\ 0 & \text{if } n \geq p+2. \end{cases}$$

We will also use the following lemma, whose proof is again omitted since it can be easily done by induction on  $r$ .

**Lemma 7** For  $d, r \in \mathbb{N}_0$  with  $d \leq r-1$ ,

$$\left( \sum_{i=d}^{r-1} \binom{i}{d}^2 \right)^{1/2} \leq \binom{r-1}{d} \left( \frac{r+d}{2d+1} \right)^{1/2} \leq \left( \frac{r}{\sqrt{3}} \right)^d r^{1/2}.$$

Finally, before proving our main result, we recall [4, Lemma 3.8].

**Lemma 8** If  $m_1, \dots, m_r \in \mathbb{N}$  and  $\sum_{i=1}^r m_i = d$ , then

$$\prod_{i=1}^r m_i \leq 3^{\min\{d, 2d-2r\}/3}.$$

We can now give the proof of our main result.

*Proof of Theorem 3:* Let  $P(X) = a_d \prod_{j=1}^r (X - v_j)^{m_j} \in \mathbb{C}[X]$  with  $v_i \neq v_j$  if  $1 \leq i < j \leq r$ ,  $m_i \in \mathbb{N}$  for  $1 \leq i \leq r$ . It is easy to see that the result holds if  $r = 1$ , so from now we suppose  $r \geq 2$ . Without loss of generality, we suppose also that  $V = \{v_1, \dots, v_r\}$  and that the roots of  $P$  are numbered in such a way that

$$|v_1| \leq \dots \leq |v_r|.$$

We give a direction to each edge in  $E$ : if  $e$  is an edge joining  $v_i$  and  $v_j$  with  $i < j$ , we consider  $e = (v_i, v_j)$  as the oriented edge going from  $v_i$  to  $v_j$ . Note that now  $G = (V, E)$  satisfies conditions 1 and 2 in Theorems 1 and 2. We consider the edges in  $E$  listed by

$$e_1 = (v_{\alpha(1)}, v_{\beta(1)}), \dots, e_{\#E} = (v_{\alpha(\#E)}, v_{\beta(\#E)}).$$

Finally, for  $1 \leq j \leq r$ , let  $d_j \in \mathbb{N}_0$  be the in-degree of the vertex  $v_j$ . Note that  $d_1 = 0$  since there is no edge finishing in  $v_1$ , and  $d_j \leq r - 1$  for  $1 \leq j \leq r$ .

As seen in [1, Section 4.2],

$$|\text{sDisc}_{d-r}(P)|^{1/2} = |a_d|^{r-1} \left( \prod_{j=1}^r m_j \right)^{1/2} \prod_{1 \leq i < j \leq r} |v_i - v_j|. \quad (1)$$

On the other hand,

$$\prod_{1 \leq i < j \leq r} |v_i - v_j| = |\det W| \quad (2)$$

where  $W$  is the Vandermonde matrix

$$W = \begin{pmatrix} 1 & v_1 & \dots & v_1^{r-1} \\ 1 & v_2 & \dots & v_2^{r-1} \\ \vdots & \vdots & & \vdots \\ 1 & v_r & \dots & v_r^{r-1} \end{pmatrix} \in \mathbb{C}^{r \times r}.$$

We consider  $F : \mathbb{C} \rightarrow \mathbb{C}^r$ ,  $F(z) = (1, z, \dots, z^{r-1})$  and define a sequence of matrices  $W_r, W_{r-1}, \dots, W_1$  in  $\mathbb{C}^{r \times r}$ . First, we define  $W_r = W$ . Then, for fixed  $j = r, \dots, 2$ , once  $W_j$  is defined, we only modify its  $j$ -th row (if any) in order to define  $W_{j-1}$ , as follows: we take the (possibly empty) sublist of edges  $e_{k_1}, \dots, e_{k_{d_j}}$  finishing in  $v_j$  and take as the  $j$ -th row of  $W_{j-1}$  the divided difference

$$F[v_{\alpha(k_1)}, \dots, v_{\alpha(k_{d_j})}, v_j].$$

Note that the  $j$ -th row of  $W_j$  equals the  $j$ -th row of  $W$ , which is  $F(v_j)$ ; and since for  $1 \leq i \leq d_j$ ,  $\alpha(k_i) < \beta(k_i) = j$ , the  $\alpha(k_i)$ -th row of  $W_j$  equals the  $\alpha(k_i)$ -th row of  $W$ , which is  $F(v_{\alpha(k_i)})$ . Then, by Lemma 5 we have that

$$\det W_j = \det W_{j-1} \prod_{i=1}^{d_j} (v_j - v_{\alpha(k_i)}).$$

In this way, we can prove by reverse induction in  $j$  that for  $j = r, \dots, 2$ ,

$$\det W = \det W_{j-1} \prod_{\substack{e \in E \\ \beta(e) \geq j}} (v_{\beta(e)} - v_{\alpha(e)}),$$

and at the end we obtain

$$\det W = \det W_1 \prod_{e \in E} (v_{\beta(e)} - v_{\alpha(e)}). \quad (3)$$

The next step is to bound  $|\det W_1|$  using Hadamard inequality. For  $1 \leq j \leq r$ , keeping the notation of the above paragraphs, the  $j$ -th row of  $W_1$  is  $F[v_{\alpha(k_1)}, \dots, v_{\alpha(k_{d_j})}, v_j]$  and by Lemma 6 its norm equals

$$\left( \sum_{i=d_j}^{r-1} \left| \sum_{\substack{(t_1, \dots, t_{d_j}, t_{d_j+1}) \in \mathbb{N}_0^{d_j+1} \\ t_1 + \dots + t_{d_j} + t_{d_j+1} = i - d_j}} \left( \prod_{l=1}^{d_j} v_{\alpha(k_l)}^{t_l} \right) v_j^{t_{d_j+1}} \right|^2 \right)^{1/2} \leq$$

$$\begin{aligned}
&\leq \left( \sum_{i=d_j}^{r-1} \binom{i}{d_j}^2 |v_j|^{2(i-d_j)} \right)^{1/2} \leq \left( \sum_{i=d_j}^{r-1} \binom{i}{d_j}^2 \right)^{1/2} \max\{1, |v_j|\}^{r-1-d_j} \leq \\
&\leq \left( \frac{r}{\sqrt{3}} \right)^{d_j} r^{1/2} \max\{1, |v_j|\}^{r-1-d_j}
\end{aligned}$$

by Lemma 7. By Hadamard inequality,

$$|\det W_1| \leq \prod_{j=1}^r \left( \frac{r}{\sqrt{3}} \right)^{d_j} r^{1/2} \max\{1, |v_j|\}^{r-1-d_j} = \left( \frac{r}{\sqrt{3}} \right)^{\#E} r^{r/2} \prod_{j=1}^r \max\{1, |v_j|\}^{r-1-d_j}. \quad (4)$$

Finally, using equations (1), (2), (3), (4) and Lemma 8,

$$\begin{aligned}
&\prod_{(v_i, v_j) \in E} |v_i - v_j| = \prod_{e \in E} |v_{\beta(e)} - v_{\alpha(e)}| = |\det W| |\det(W_1)|^{-1} \geq \\
&\geq |\text{sDisc}_{d-r}(P)|^{1/2} |a_d|^{-(r-1)} \left( \prod_{j=1}^r \max\{1, |v_j|\}^{-(r-1-d_j)} \right) \left( \frac{r}{\sqrt{3}} \right)^{-\#E} r^{-r/2} \left( \prod_{j=1}^r m_j \right)^{-1/2} \geq \\
&\geq |\text{sDisc}_{d-r}(P)|^{1/2} M(P)^{-(r-1)} \left( \frac{r}{\sqrt{3}} \right)^{-\#E} r^{-r/2} \left( \frac{1}{3} \right)^{\min\{d, 2d-2r\}/6}
\end{aligned}$$

as we wanted to prove.  $\square$

We include below some remarks considering cases in which the bound in Theorem 3 can be slightly improved.

**Remark 9** *Following the notation in Theorem 3, for  $1 \leq j \leq r$  let  $\tilde{d}_j$  be the total degree of vertex  $v_j$  and let  $\tilde{d} = \min\{\tilde{d}_j \mid 1 \leq j \leq r\}$ . If  $P$  is a monic polynomial then*

$$\prod_{(v_i, v_j) \in E} |v_i - v_j| \geq |\text{sDisc}_{d-r}(P)|^{1/2} M(P)^{-(r-1-\frac{1}{2}\tilde{d})} \left( \frac{r}{\sqrt{3}} \right)^{-\#E} r^{-r/2} \left( \frac{1}{3} \right)^{\min\{d, 2d-2r\}/6}.$$

Indeed, taking into account that  $|v_{\alpha(e)}| \leq |v_{\beta(e)}|$  for every  $e \in E$ , we change the last part of the proof of Theorem 3 as follows:

$$\begin{aligned}
&\prod_{(v_i, v_j) \in E} |v_i - v_j| = \prod_{e \in E} |v_{\beta(e)} - v_{\alpha(e)}| = |\det W| |\det(W_1)|^{-1} \geq \\
&\geq |\text{sDisc}_{d-r}(P)|^{1/2} \left( \prod_{j=1}^r \max\{1, |v_j|\}^{-(r-1-d_j)} \right) \left( \prod_{e \in E} \frac{\max\{1, |v_{\alpha(e)}|\}^{1/2}}{\max\{1, |v_{\beta(e)}|\}^{1/2}} \right) \left( \frac{r}{\sqrt{3}} \right)^{-\#E} r^{-r/2} \left( \prod_{j=1}^r m_j \right)^{-1/2} \geq \\
&\geq |\text{sDisc}_{d-r}(P)|^{1/2} \left( \prod_{j=1}^r \max\{1, |v_j|\}^{-(r-1-\frac{1}{2}\tilde{d}_j)} \right) \left( \frac{r}{\sqrt{3}} \right)^{-\#E} r^{-r/2} \left( \frac{1}{3} \right)^{\min\{d, 2d-2r\}/6} \geq \\
&\geq |\text{sDisc}_{d-r}(P)|^{1/2} M(P)^{-(r-1-\frac{1}{2}\tilde{d})} \left( \frac{r}{\sqrt{3}} \right)^{-\#E} r^{-r/2} \left( \frac{1}{3} \right)^{\min\{d, 2d-2r\}/6}.
\end{aligned}$$

The next remark considers the case where a number of small distances is guaranteed by some extra information (possibly coming from numerical computations). It could be particularly useful to bound the minimal distance between different roots, taking  $E$  as the set with only one edge joining a pair of closest roots.

**Remark 10** *Following the notation in Theorem 3, suppose that  $r > 2$  and that there exist at least  $k$  distinct pairs of roots  $(v_{\gamma(1)}, v_{\delta(1)}), \dots, (v_{\gamma(k)}, v_{\delta(k)})$  whose distance is less than  $\frac{\sqrt{3}}{r}$  (not necessarily these pairs of roots should be connected by edges in  $E$ ). For  $1 \leq i \leq k$ , let  $\Delta_i > 0$  such that*

$$|v_{\gamma(i)} - v_{\delta(i)}| \leq \left(\frac{\sqrt{3}}{r}\right)^{1+\Delta_i}$$

and renumber these pairs such that

$$\Delta_1 \geq \dots \geq \Delta_k.$$

Then, if  $\#E < k$ ,

$$\prod_{(v_i, v_j) \in E} |v_i - v_j| \geq |\text{sDisc}_{d-r}(P)|^{1/2} M(P)^{-(r-1)} \left(\frac{r}{\sqrt{3}}\right)^{-\#E + \Delta_{\#E+1} + \dots + \Delta_k} r^{-r/2} \left(\frac{1}{3}\right)^{\min\{d, 2d-2r\}/6}.$$

Indeed, suppose that

$$0 < \omega_1 \leq \dots \leq \omega_{\binom{r}{2}}$$

are the ordered distances between pairs of roots of  $P$ . By the assumptions, for  $1 \leq i \leq k$  there are at least  $i$  distances less than or equal to  $\left(\frac{\sqrt{3}}{r}\right)^{1+\Delta_i}$  and then we have that  $\omega_i \leq \left(\frac{\sqrt{3}}{r}\right)^{1+\Delta_i}$ . Consider  $\tilde{E}$  the set of  $k$  edges whose lengths are  $\omega_1, \dots, \omega_k$ . Then, applying the bound in Theorem 3 to  $G = (V, \tilde{E})$  we obtain

$$\begin{aligned} \prod_{(v_i, v_j) \in E} |v_i - v_j| &\geq \prod_{i=1}^{\#E} \omega_i = \left( \prod_{(v_i, v_j) \in \tilde{E}} |v_i - v_j| \right) \left( \prod_{i=\#E+1}^k \omega_i^{-1} \right) \geq \\ &\geq |\text{sDisc}_{d-r}(P)|^{1/2} M(P)^{-(r-1)} \left(\frac{r}{\sqrt{3}}\right)^{-k} r^{-r/2} \left(\frac{1}{3}\right)^{\min\{d, 2d-2r\}/6} \left( \prod_{i=\#E+1}^k \left(\frac{r}{\sqrt{3}}\right)^{1+\Delta_i} \right) = \\ &= |\text{sDisc}_{d-r}(P)|^{1/2} M(P)^{-(r-1)} \left(\frac{r}{\sqrt{3}}\right)^{-\#E + \Delta_{\#E+1} + \dots + \Delta_k} r^{-r/2} \left(\frac{1}{3}\right)^{\min\{d, 2d-2r\}/6}. \end{aligned}$$

Finally, as an application of Theorem 3, we give a simplified proof of [7, Theorem 9] with smaller constants.

**Theorem 11** *Let  $P \in \mathbb{C}[X]$  be a polynomial of degree  $d$  with exactly  $r \geq 2$  distinct complex roots and let  $V = \{v_1, \dots, v_r\} \subset \mathbb{C}$  be the set of roots. For any root  $v$  of  $P$ , we denote by  $\text{sep}(P, v)$  the distance from  $v$  to (one of) its closest different root of  $P$ . Then, for any  $V' \subset V$ ,*

$$\prod_{v \in V'} \text{sep}(P, v) \geq |\text{sDisc}_{d-r}(P)| M(P)^{-2(r-1)} \left(\frac{r}{\sqrt{3}}\right)^{-\#V'} r^{-r} \left(\frac{1}{3}\right)^{\min\{d, 2d-2r\}/3}.$$

*Proof:* For each  $v \in V$ , we take  $\tilde{v}$  as (one of) its closest different root of  $P$ . We consider the multigraph  $G = (V, E)$  where  $E$  is the multiset of edges of type  $(v, \tilde{v})$  with  $v \in V'$ . Note that each edge in  $E$  can occur at most 2 times (one for each of its vertex). We divide  $E$  in two sets  $E_0$  and  $E_1$ , with  $E_0$  having all the

elements in  $E$  and  $E_1$  having the elements that occur twice in  $E$ . Applying Theorem 3 to  $(V, E_0)$  and  $(V, E_1)$  and taking into account that  $\#E_0 + \#E_1 = \#V'$ , we obtain

$$\begin{aligned} \prod_{v \in V'} \text{sep}(P, v) &= \left( \prod_{(v_i, v_j) \in E_0} |v_i - v_j| \right) \left( \prod_{(v_i, v_j) \in E_1} |v_i - v_j| \right) \geq \\ &\geq |\text{sDisc}_{d-r}(P)| M(P)^{-2(r-1)} \left( \frac{r}{\sqrt{3}} \right)^{-\#V'} r^{-r} \left( \frac{1}{3} \right)^{\min\{d, 2d-2r\}/3} \end{aligned}$$

as we wanted to prove. □

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